

HOMOGENEOUS HYPER-HERMITIAN METRICS WHICH ARE CONFORMALLY HYPER-KÄHLER

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ABSTRACT. Let g be a hyper-Hermitian metric on a simply connected hypercomplex four-manifold (M, \mathcal{H}) . We show that when the isometry group $I(M, g)$ contains a subgroup acting simply transitively on M by hypercomplex isometries then the metric g is conformal to a hyper-Kähler metric. We describe explicitly the corresponding hyper-Kähler metrics and it follows that, in four dimensions, these are the only hyper-Kähler metrics containing a homogeneous metric in its conformal class.

1. PRELIMINARIES

A hypercomplex structure on a $4n$ -dimensional manifold M is a family $\mathcal{H} = \{J_\alpha\}_{\alpha=1,2,3}$ of fibrewise endomorphisms of the tangent bundle TM of M satisfying:

$$(1.1) \quad J_\alpha^2 = -I, \quad \alpha = 1, 2, 3 \quad J_1 J_2 = -J_2 J_1 = J_3,$$

$$(1.2) \quad N_\alpha \equiv 0, \quad \alpha = 1, 2, 3$$

where I is the identity on the tangent space $T_p M$ of M at p for all p in M and N_α is the Nijenhuis tensor corresponding to J_α :

$$N_\alpha(X, Y) = [J_\alpha X, J_\alpha Y] - [X, Y] - J_\alpha([X, J_\alpha Y] + [J_\alpha X, Y])$$

for all X, Y vector fields on M . A differentiable map $f : M \rightarrow M$ is said to be hypercomplex if it is holomorphic with respect to J_α , $\alpha = 1, 2, 3$. The group of hypercomplex diffeomorphisms on (M, \mathcal{H}) will be denoted by $\text{Aut}(\mathcal{H})$.

A riemannian metric g on a hypercomplex manifold (M, \mathcal{H}) is called hyper-Hermitian when $g(J_\alpha X, J_\alpha Y) = g(X, Y)$ for all vectors fields X, Y on M , $\alpha = 1, 2, 3$.

Given a manifold M with a hypercomplex structure $\mathcal{H} = \{J_\alpha\}_{\alpha=1,2,3}$ and a hyper-Hermitian metric g consider the 2-forms ω_α , $\alpha = 1, 2, 3$, defined by

$$(1.3) \quad \omega_\alpha(X, Y) = g(X, J_\alpha Y).$$

The metric g is said to be hyper-Kähler when $d\omega_\alpha = 0$ for $\alpha = 1, 2, 3$.

It is well known that a hyper-Hermitian metric g is conformal to a hyper-Kähler metric \tilde{g} if and only if there exists an exact 1-form $\theta \in \Lambda^1 M$ such that

$$(1.4) \quad d\omega_\alpha = \theta \wedge \omega_\alpha, \quad \alpha = 1, 2, 3$$

where, if $g = e^f \tilde{g}$ for some $f \in C^\infty(M)$, then $\theta = df$.

We prove the following result:

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Theorem 1.1. *Let (M, \mathcal{H}, g) be a simply connected hyper-Hermitian 4-manifold. Assume that there exists a Lie group $G \subset I(M, g) \cap \text{Aut}(\mathcal{H})$ acting simply transitively on M . Then g is conformally hyper-Kähler.*

We conclude that one of the hyper-Kähler metrics constructed by the Gibbons-Hawking ansatz [4] contains a homogeneous hyper-Hermitian metric in its conformal class. This hyper-Hermitian metric is not symmetric and has negative sectional curvature [1].

As a consequence of Theorem 1.1 and the results in [1] we obtain that the following symmetric riemannian metrics are conformally hyper-Kähler:

- the riemannian product of the canonical metrics on $\mathbb{R} \times S^3$;
- the riemannian product of the canonical metrics on $\mathbb{R} \times \mathbb{R}H^3$, where $\mathbb{R}H^3$ denotes the real hyperbolic space;
- the canonical metric on the real hyperbolic space $\mathbb{R}H^4$.

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2. PROOF OF THE MAIN THEOREM

Proof of Theorem 1.1. Since G acts simply transitively on M then M is diffeomorphic to G and therefore the hypercomplex structure and hyper-Hermitian metric can be transferred to G and will also be denoted by $\{J_\alpha\}_{\alpha=1,2,3}$ and g , respectively. Since G acts by hypercomplex isometries it follows that both $\{J_\alpha\}_{\alpha=1,2,3}$ and g are left invariant on G . All such simply connected Lie groups were classified in [1], where it is shown that the Lie algebra \mathfrak{g} of G is either abelian or isomorphic to one of the following Lie algebras (we fix an orthonormal basis $\{e_j\}_{j=1,\dots,4}$ of \mathfrak{g}):

1. $[e_2, e_3] = e_4$, $[e_3, e_4] = e_2$, $[e_4, e_2] = e_3$, e_1 central;
2. $[e_1, e_3] = e_1$, $[e_2, e_3] = e_2$, $[e_1, e_4] = e_2$, $[e_2, e_4] = -e_1$;
3. $[e_1, e_j] = e_j$, $j = 2, 3, 4$;
4. $[e_3, e_4] = \frac{1}{2}e_2$, $[e_1, e_2] = e_2$, $[e_1, e_j] = \frac{1}{2}e_j$, $j = 3, 4$.

Observe that in case 1 above M is diffeomorphic to $\mathbb{R} \times S^3$ while in the remaining cases it is diffeomorphic to \mathbb{R}^4 , therefore in all cases any closed form on M is exact. We now proceed by finding in each case a closed form $\theta \in \Lambda^1 \mathfrak{g}^*$ satisfying (1.4). Note that we work on the Lie algebra level since g and ω_α are all left invariant on G . Let $\{e^j\}_{j=1,\dots,4} \subset \Lambda^1 \mathfrak{g}^*$ be the dual basis of $\{e_j\}_{j=1,\dots,4}$. From now on we will write $e^{ij\dots}$ to denote $e^i \wedge e^j \wedge \dots$. In all the cases below the 2-forms ω_α are determined from (1.3) in terms of the hypercomplex structures constructed in [1].

Case 1. The 2-forms ω_α are given as follows:

$$\omega_1 = -e^{12} - e^{34}, \quad \omega_2 = -e^{13} + e^{24}, \quad \omega_3 = -e^{14} - e^{23}.$$

To calculate $d\omega_\alpha$ we obtain first de^j (recall that $d\sigma(x, y) = -\sigma[x, y]$ for $\sigma \in \Lambda^1 \mathfrak{g}^*$):

$$(2.1) \quad de^1 = 0, \quad de^2 = -e^{34}, \quad de^3 = e^{24}, \quad de^4 = -e^{23}.$$

These equations and the fact that $d(\sigma \wedge \tau) = d\sigma \wedge \tau + (-1)^r \sigma \wedge d\tau$ for all $\sigma \in \Lambda^r \mathfrak{g}^*$ give the following formulas:

$$d\omega_1 = -e^{134}, \quad d\omega_2 = e^{124}, \quad d\omega_3 = -e^{123}$$

from which we conclude that (1.4) holds for $\theta = e^1$, which is closed and therefore exact since G is diffeomorphic to $\mathbb{R} \times S^3$. We conclude that this hyper-Hermitian metric, which, as shown in [1], is homothetic to the riemannian product of the canonical metrics on $\mathbb{R} \times S^3$, is conformal to a hyper-Kähler metric.

Case 2. In this case we have the following equations for ω_α :

$$\omega_1 = e^{14} - e^{23}, \quad \omega_2 = -e^{12} + e^{34}, \quad \omega_3 = -e^{13} - e^{24}.$$

and we calculate

$$(2.2) \quad de^1 = -e^{13} + e^{24}, \quad de^2 = -e^{23} - e^{14}, \quad de^3 = 0, \quad de^4 = 0,$$

$$(2.3) \quad d\omega_1 = -2e^{134}, \quad d\omega_2 = -2e^{123}, \quad d\omega_3 = 2e^{234}$$

so that (1.4) is satisfied for $\theta = 2e^3$, which again is closed, so this hyper-Hermitian metric is also conformal to a hyper-Kähler metric. In this case the hyper-Hermitian metric is homothetic to the riemannian product of the canonical metrics on $\mathbb{R} \times \mathbb{R}H^3$, where $\mathbb{R}H^3$ denotes the real hyperbolic space.

Case 3. In this case the 2-forms ω_α are given as follows:

$$\omega_1 = -e^{12} - e^{34}, \quad \omega_2 = -e^{13} + e^{24}, \quad \omega_3 = -e^{14} - e^{23}$$

and a calculation of exterior derivatives gives:

$$(2.4) \quad de^1 = 0, \quad de^j = -e^{1j}, \quad j = 2, 3, 4$$

$$(2.5) \quad d\omega_1 = 2e^{134}, \quad d\omega_2 = -2e^{124}, \quad d\omega_3 = -2e^{123}$$

so that (1.4) is satisfied for $\theta = -2e^1$. This hyper-Hermitian metric is homothetic to the canonical metric on the real hyperbolic space $\mathbb{R}H^4$.

Case 4. In this case we have the following equations for ω_α :

$$\omega_1 = -e^{12} + e^{34}, \quad \omega_2 = -e^{13} - e^{24}, \quad \omega_3 = e^{14} - e^{23}$$

and we calculate

$$(2.6) \quad de^1 = 0, \quad de^2 = -e^{12} - \frac{1}{2}e^{34}, \quad de^j = -\frac{1}{2}e^{1j}, \quad j = 3, 4$$

$$(2.7) \quad d\omega_1 = -\frac{3}{2}e^{134}, \quad d\omega_2 = \frac{3}{2}e^{124}, \quad d\omega_3 = \frac{3}{2}e^{123}$$

so that (1.4) is satisfied for $\theta = -\frac{3}{2}e^1$. This hyper-Hermitian metric is not symmetric and has negative sectional curvature (cf. [1]).

Remark 2.1. All the hyper-Hermitian manifolds (M, \mathcal{H}, g) considered above admit a connection ∇ such that:

$$\nabla g = 0, \quad \nabla J_\alpha = 0, \quad \alpha = 1, 2, 3$$

and the $(3, 0)$ tensor $c(X, Y, Z) = g(X, T(Y, Z))$ is totally skew-symmetric, where T is the torsion of ∇ . Such a connection is called an HKT connection (cf. [5]). In case M is diffeomorphic to $\mathbb{R} \times S^3$ it can be shown that, moreover, the corresponding 3-form c is closed.

3. COORDINATE DESCRIPTION OF THE HYPER-KÄHLER METRICS

In this section we will use global coordinates on each of the Lie groups considered in the previous section to describe the corresponding hyper-Kähler metrics. This will allow us to identify the hyper-Kähler metric in §2, Case 4, with one constructed by the Gibbons-Hawking ansatz [4].

$$\text{Case 1. } G = \mathbb{H}^* = GL(1, \mathbb{H}) = \left\{ \begin{pmatrix} x & -y & -z & -t \\ y & x & -t & z \\ z & t & x & -y \\ t & -z & y & x \end{pmatrix} : (x, y, z, t) \in \mathbb{R}^4 - \{0\} \right\}.$$

We obtain a basis of left invariant 1-forms on G as follows. Set $r^2 = x^2 + y^2 + z^2 + t^2$, $r > 0$, and $\Omega = g^{-1}dg$ for $g \in G$, that is,

$$\text{if } g = \begin{pmatrix} x & -y & -z & -t \\ y & x & -t & z \\ z & t & x & -y \\ t & -z & y & x \end{pmatrix} \text{ then } \Omega = \begin{pmatrix} \sigma_1 & -\sigma_2 & -\sigma_3 & -\sigma_4 \\ \sigma_2 & \sigma_1 & -\sigma_4 & \sigma_3 \\ \sigma_3 & \sigma_4 & \sigma_1 & -\sigma_2 \\ \sigma_4 & -\sigma_3 & \sigma_2 & \sigma_1 \end{pmatrix}$$

where

$$\begin{pmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \end{pmatrix} = \frac{1}{r^2} \begin{pmatrix} x & y & z & t \\ -y & x & t & -z \\ -z & -t & x & y \\ -t & z & -y & x \end{pmatrix} \begin{pmatrix} dx \\ dy \\ dz \\ dt \end{pmatrix}.$$

Then σ_j , $1 \leq j \leq 4$, is a basis of left invariant 1-forms on G and it follows from $d\Omega + \Omega \wedge \Omega = 0$ that

$$d\sigma_1 = 0, \quad d\sigma_2 = -2\sigma_3 \wedge \sigma_4, \quad d\sigma_3 = 2\sigma_2 \wedge \sigma_4, \quad d\sigma_4 = -2\sigma_2 \wedge \sigma_3.$$

Setting

$$e^1 = 2\sigma_1, \quad e^2 = 2\sigma_2, \quad e^3 = 2\sigma_3, \quad e^4 = 2\sigma_4,$$

so that $\{e^j\}_{1 \leq j \leq 4}$ satisfy (2.1), the left-invariant hyper-Hermitian metric is

$$(3.1) \quad g = (e^1)^2 + (e^2)^2 + (e^3)^2 + (e^4)^2 = \frac{4}{r^2}(dx^2 + dy^2 + dz^2 + dt^2)$$

that is, g is the standard conformally flat metric on $\mathbb{R}^4 - \{0\}$, and since the Lee form is $\theta = e^1 = d(2 \log r)$ the corresponding hyper-Kähler metric is $\tilde{g} = e^{-2 \log r} g$, that is,

$$(3.2) \quad \tilde{g} = \frac{4}{r^2} \left(\frac{(dr)^2}{r^2} + (\sigma_2)^2 + (\sigma_3)^2 + (\sigma_4)^2 \right) = \frac{4}{r^4}(dx^2 + dy^2 + dz^2 + dt^2).$$

Observe that the standard metric on any coordinate quaternionic Hopf surface is locally conformally equivalent to \tilde{g} (cf. [2]).

Case 2. Define a product on \mathbb{R}^4 as follows:

$$(x, y, z, t)(x', y', z', t') = (x + e^z(x' \cos t - y' \sin t), y + e^z(x' \sin t + y' \cos t), z + z', t + t').$$

This defines a Lie group structure on \mathbb{R}^4 that makes it isomorphic to the Lie group considered in §2, Case 2. The following 1-forms are left-invariant with respect to

the above product:

$$(3.3) \quad e^1 = e^{-z} \cos t \, dx + e^{-z} \sin t \, dy, \quad e^3 = -dz,$$

$$(3.4) \quad e^2 = -e^{-z} \sin t \, dx + e^{-z} \cos t \, dy, \quad e^4 = -dt$$

These forms satisfy relations (2.2). The hyper-Hermitian metric is therefore given as follows:

$$g = (e^1)^2 + (e^2)^2 + (e^3)^2 + (e^4)^2 = e^{-2z}(dx^2 + dy^2) + dz^2 + dt^2$$

and the Lee form is $\theta = 2e^3 = -2dz$, so that the hyper-Kähler metric becomes

$$\tilde{g} = e^{2z}g = (dx^2 + dy^2) + e^{2z}(dz^2 + dt^2).$$

Observe that the change of coordinates $s = e^z$ gives the following simple form for \tilde{g} on $\mathbb{R}^+ \times \mathbb{R}^3$:

$$\tilde{g} = dx^2 + dy^2 + (ds^2 + s^2 dt^2).$$

This allows us to identify \tilde{g} with the riemannian product of two Kähler metrics: the euclidean metric on \mathbb{R}^2 with the warped product cone metric on $\mathbb{R}^+ \times \mathbb{R}$ (cf. [3]).

Case 3. We endow \mathbb{R}^4 with the following product:

$$(x, y, z, t)(x', y', z', t') = (x + e^t x', y + e^t y', z + e^t z', t + t')$$

thereby obtaining the Lie group structure considered in §2, Case 3, with corresponding left-invariant 1-forms:

$$e^1 = dt, \quad e^2 = e^{-t} dx, \quad e^3 = e^{-t} dy, \quad e^4 = e^{-t} dz.$$

The hyper-Hermitian metric is therefore

$$g = (e^1)^2 + (e^2)^2 + (e^3)^2 + (e^4)^2 = e^{-2t}(dx^2 + dy^2 + dz^2) + dt^2$$

with corresponding Lee form $\theta = -2e^1 = -2dt$, yielding the following hyper-Kähler metric:

$$\tilde{g} = e^{2t}g = dx^2 + dy^2 + dz^2 + e^{2t}dt^2.$$

Setting $s = e^t$, \tilde{g} is the euclidean metric $ds^2 + dx^2 + dy^2 + dz^2$ on $\mathbb{R}^+ \times \mathbb{R}^3$.

Case 4. Consider the following product on \mathbb{R}^4 :

$$(x, y, z, t)(x', y', z', t') = (x + e^{\frac{t}{2}} x', y + e^{\frac{t}{2}} y', z + e^t z' + \frac{e^{\frac{t}{2}}}{4}(xy' - yx'), t + t')$$

which yields the Lie group structure considered in §2, Case 4. It is easily checked that the following left-invariant 1-forms satisfy (2.6):

$$e^1 = dt, \quad e^2 = e^{-t}(dz - \frac{1}{4}xdy + \frac{1}{4}ydx), \quad e^3 = e^{-\frac{t}{2}}dx, \quad e^4 = e^{-\frac{t}{2}}dy.$$

The hyper-Hermitian metric is now obtained as in the above cases:

$$\begin{aligned} g &= (e^1)^2 + (e^2)^2 + (e^3)^2 + (e^4)^2 \\ &= dt^2 + e^{-t}(dx^2 + dy^2) + e^{-2t}(dz - \frac{1}{4}(xdy - ydx))^2 \end{aligned}$$

and the Lee form is $\theta = -\frac{3}{2}dt$, from which we obtain the hyper-Kähler metric as usual:

$$\tilde{g} = e^{-\frac{3}{2}t}dt^2 + e^{-\frac{t}{2}}(dx^2 + dy^2) + e^{-\frac{t}{2}}(dz - \frac{1}{4}(xdy - ydx))^2.$$

Setting $s = e^{\frac{t}{2}}$, \tilde{g} becomes

$$\tilde{g} = s(ds^2 + dx^2 + dy^2) + \frac{1}{s}(dz - \frac{1}{4}(xdy - ydx))^2$$

on $\mathbb{R}^+ \times \mathbb{R}^3$, which allows us to identify \tilde{g} with one of the hyper-Kähler metrics constructed by the Gibbons-Hawking ansatz [4]. The identification is easily obtained from [6], Proposition 1.

We can now rephrase Theorem 1.1 as follows, where $[h]$ denotes the conformal class of h :

Corollary 3.1. *Let h be a hyper-Kähler metric on a simply connected hypercomplex 4-manifold (M, \mathcal{H}) such that there exist $g \in [h]$ and a Lie group $G \subset I(M, g) \cap \text{Aut}(\mathcal{H})$ acting simply transitively on M . Then (M, h) is homothetic to either \mathbb{R}^4 with the euclidean metric or one of the following riemannian manifolds:*

1. $M = \mathbb{R}^4 - \{0\}$, $h = r^{-4}(dx^2 + dy^2 + dz^2 + dt^2)$,
2. $M = \mathbb{R}^2 \times (\mathbb{R}^+ \times \mathbb{R})$, $h = (dx^2 + dy^2) + (ds^2 + s^2 dt^2)$,
3. $M = \mathbb{R}^+ \times \mathbb{R}^3$, $h = ds^2 + dx^2 + dy^2 + dz^2$,
4. $M = \mathbb{R}^+ \times \mathbb{R}^3$, $h = s(ds^2 + dx^2 + dy^2) + s^{-1}(dz - \frac{1}{4}(xdy - ydx))^2$.

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